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Some Generalized Differentiation Formulae for $\overline{H}$ -Function

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Abstract: **This paper contains differentiation formulae for H-function. The generalize results established by Devra and Raithie [5; 107-113] special cases include known and new formulae for various special functions such as H̄--function, generalized Wright hypergeometric function. Mathematics Subject Classification: 33A20; 26A33**

Keywords: ***$\overline{H}$*-function, H-function, Wright hypergeometric function**.

I. INTRODUCTION

Inayat -Hussain (1987a) introduced generalization form of Fox's H̄-function which is popularly known as H̄ −function.

Now H̄-function stands on fairly firm footing through the research contribution of various authors like Inayat- Hussain (1987b), Rathie (1993), Gupta & Soni (2005), Chaurasia & singh (2010), Agrawal & Mehar (2012), Marko, Pandey & Sukla (2013).

The H̄-Function is defined and represented in the following manner [2; 10].

$$\overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left[z\Big|_{(b_j,B_j)_{1,m}\ ,\ (b_j,B_j;\beta_j)_{m+1,q}}^{(a_j,A_j;\alpha_j)_{1,n}\ ,\ (a_j,A_j)_{n+1,p}}\right] = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \theta(s)\, z^s\, ds. \qquad (1)$$

where $\theta(s)$ is given by

$$\theta(s) = \frac{\Pi_{j=1}^{m}\Gamma(b_j - B_j s)\ \Pi_{j=1}^{n}\Gamma^{\alpha_j}(1 - a_j + A_j s)}{\Pi_{j=m+1}^{q}\Gamma^{\beta_j}(1 - b_j + B_j s)\ \Pi_{j=n+1}^{p}\Gamma(a_j - A_j s)}. \qquad (2)$$

Also
(i) $z \neq 0$.
(ii) $i = \sqrt{(-1)}$.
(iii) m,n,p,q are integers satisfying $0 \leq m \leq q$, $0 \leq n \leq p$.
(iv) L is a suitable contour in the complex plane.
(v) An empty product is interpreted as unity.
(vi) $A_j, j = 1,...,p; B_j, j = 1,...,q; \alpha_j, j = 1,...,n; \beta_j, j = 1,...,q$ are real positive numbers.
(vii) $a_j, j = 1,...p$ and $b_j, j = 1,...,q$ are all complex numbers.

The nature of contour L, sufficient conditions of convergence of defining integral (1) and other details about the H̄-Function can be seen in the paper [8].

The behaviour of the H̄-function for small values of jzj follows easily from a result given by [12]:

$$\overline{H}_{p,q}^{m,n}[z] = O(|z|^{\alpha});$$

where

$$\alpha = \min_{1\le j\le m} Re(\frac{b_j}{\alpha_j}), |z| \to 0.$$

$$\Omega = \sum_{j=1}^{m} |\beta_j| + \sum_{j=1}^{n} |\alpha_j A_j| - \sum_{j=m+1}^{q} |\beta_j B_j| - \sum_{j=n+1}^{p} A_j > 0 \; and \; 0 < |z| < \infty.$$

If $f(x)$ is a linear polynomial in x and D represents $\frac{d}{dx}$ then

$$f(D\,x)\; x^{\alpha} = f(\alpha+1)\; x^{\alpha} \tag{3}$$

The ${}_p\overline{\psi}_q(z)$ function will be defined and represented [8] as follows

$${}_p\overline{\psi}_q[^{(a_j,A_j;\alpha_j)_{1,p}}_{(b_j,B_j;\beta_j)_{1,q}}; z] = \overline{H}^{1,p}_{p,q+1}[-z/^{(1-a_j,A_j;\alpha_j)_{1,p}}_{(0,1),(1-b_j,B_j;\beta_j)_{1,q}}] \tag{4}$$

The function ${}_p\overline{\psi}_q(z)$ is termed as generalized Wright hypergeometric function because it gives ${}_p\psi_q$ for $A_j = 1, j = 1, ..., p; B_j = 1, j = 1, ..., q$ in it.

An important special case of ${}_p\overline{\psi}_q(z)$ that generalizes several special functions of practical importance is given as

$${}_p\overline{F}_q[^{(a_j,1;\alpha_j)_{1,p}}_{(b_j,1;\beta_j)_{1,q}}; z] = \frac{\prod_{j=1}^{q}\{\Gamma(b_j)\}^{\beta_j}}{\prod_{j=1}^{p}\{\Gamma(a_j)\}^{\alpha_j}} \overline{H}^{1,p}_{p,q+1}[-z/^{(1-a_j,1;\alpha_j)_{1,p}}_{(0,1),(1-b_j,1;\beta_j)_{1,q}}] \tag{5}$$

The function ${}_p\overline{\psi}_q(z)$ reduces to the well known ${}_p\psi_q(z)$ for $\alpha_j = 1, ..., p; \beta_j = 1, j = 1, ..., q$ in it.
In this paper for the sake of brevity we shall use the following contracted notation for $\overline{H}$-function in (1).
where
A stand for $(a_j, A_j; \alpha_j)_{1,n}$, **B** stand for $(b_j, B_j)_{1,m}$
C stand for $(a_j, A_j)_{n+1,p}$, **D** stand for $(b_j, B_j; \beta_j)_{m+1,q}$.

II. MAIN RESULTS

$$[D(ax^v+b)-\lambda_1]...[D(ax^v+b)-\lambda_r]\{(ax^v+b)^{\alpha}\overline{H}^{m,n}_{p,q}[z(ax^v+b)^h/^{\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D}}]\}$$

$$= a^r(ax^v+b)^{\alpha}\overline{H}^{m,n+r}_{p+r,q+r}[z(ax^v+b)^h/^{(c_j-\alpha-1,h;1)_{1,r},\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D},(c_j-\alpha,h;1)_{1,r}}] \tag{6}$$

provided $h > 0$ and a, b are complex numbers, r is a positive integers $.\lambda_i = ac_i$ where a and c_i are not simultaneously zero, and $i = 1, .., r$.

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Proof: Taking L.H.S. of (6) and using definition of H̄-function (1) and (2), operating under the integral sign using (3),the expression becomes

$$a^r(ax^v+b)^\alpha \frac{1}{2\pi i}\int_L \theta(s)[\alpha+hs+1-c_1]...[\alpha+hs+1-c_r]z^s(ax^v+b)^{hs}ds.$$

Expressing $(\alpha+hs+1-c_j)$as

$$(\alpha+hs+1-c_j)=\frac{\Gamma(\alpha+hs+2-c_j)}{(\alpha+hs+1-c_j)}, forj=1,...,r$$

and interpreting with the help of (1), the result follows.

Differentiation under the integral sign used in the proof is valid, provided

(i) $\theta(s)z^s(ax^v+b)^{\alpha+hs}$ is continuous function of x and s.
(ii) $\theta(s)(\alpha+hs+1-c_1)...(\alpha+hs+1-c_r)z^s(ax^v+b)\alpha+hs$ is continuous function of x and s.
(iii) $\int_L \theta(s)z^s(ax^v+b)^{hs}ds$ converges.
(iv) $\int_L \theta(s)(\alpha+hs+1-c_1)...(\alpha+hs+1-c_r)z^s(ax^v+b)^{\alpha+hs}\,\alpha^{hs}$ is converges uniformly with respect to x.

The conditions on continuity are clearly satisfied and conditions on convergence are satisfied when the L.H.S. of (6) exists.

Special case: When the λ's are in the arithmetic progression (6) takes the following form-

$$[D(ax^v+b)-\lambda][D(ax^v+b)-\lambda+k]...[D(ax^v+b)-\lambda+(r-1)k]$$

$$.\{(ax^v+b)^{\alpha d+c-1}\overline{H}^{m,n}_{p,q}[z(ax^v+b)^{hd}/^{\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D}}]\}$$

$$=a^rd^r(ax^v+b)^{\alpha d+c-1}\overline{H}^{m,n+1}_{p+1,q+1}[z(ax^v+b)^{hd}/^{(1-r-\alpha,h;1),\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D},(1-\alpha,h;1)}] \tag{7}$$

provided $d\neq 0, h>0$ a and b are complex number, r is a positive integer, $\lambda=ac, k=ad$, a and c are not simultaneously zero.

Proof: Taking L.H.S. of (7) and using (1), where $\theta(s)$ is given by (2), we have operating under the integral sign using (3), the expression becomes

$$a^rd^r(ax^v+b)^{2d+c-1}\frac{1}{2\pi i}\int_L \theta(s)[\alpha+hs+hs][\alpha+hs+1]$$

$$...[\alpha+hs+r-1]z^s(ax^v+b)^{hds}ds.$$

Expressing

$$(\alpha+hs)(\alpha+hs+1)...(\alpha+hs+r-1)=\frac{\Gamma(\alpha+hs+r)}{\Gamma(\alpha+hs)}$$

and interpreting with the help of (1), the result follows when $\lambda_1=\lambda_2=\lambda_3=....=\lambda_r=\lambda$ (6) becomes

$$[D(ax^v+b)-\lambda]^r\{(ax^v+b)^{\alpha+c-1}.\overline{H}^{m,n}_{p,q}[z(ax^v+b)^h/^{\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D}}]\}$$

$$=a^r(ax^v+b)^{\alpha+c-1}\overline{H}^{m,n+r}_{p+r,q+r}[z(ax^v+b)^h/^{(-\alpha,h;1)_{1,r},\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D},(1-\alpha,h;1)_{1,r}}] \tag{8}$$

provided $h > 0$, a and b are complex numbers, r is a positive integer.

when $c = 0$, (8) reduces to

$$[D(ax^v + b) - \lambda]^r\{(ax^v + b)^{\alpha-1}.\overline{H}^{m,n}_{p,q}[z(ax^v + b)^h/^{\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D}}]\}$$

$$= a^r(ax^v + b)^{\alpha-1}\overline{H}^{m,n+r}_{p+r,q+r}[z(ax^v + b)^h/^{(-\alpha,h;1)_{1,r},\,\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D},\,(1-\alpha,h;1)_{1,r}}] \tag{9}$$

provided $h > 0$, a and b are complex numbers, r is a positive integer.

Specializing the parameters in (6) and using 4, (6) reduces to

$$[D(ax^v + b) - \lambda_1]...[D(ax^v + b) - \lambda_r]\{(ax^v + b)^{\alpha}.{}_p\overline{\psi}_q[^{(a_j,A_j;\alpha_j)_{1,p}}_{(b_j,B_j;\beta_j)_{1,q}}; z(ax^v + b)^h]\}$$

$$= a^r(ax^v + b)^{\alpha}_{p+r}\overline{\psi}_{q+r}[/^{(\alpha-c_j+2,h;1)_{1,r},\,(a_j,A_j;\alpha_j)_{1,p}}_{(1+\alpha_j-c_j,h;1)_{1,r},\,(b_j,B_j;\beta_j)_{1,q}}] \tag{10}$$

provided $h > 0$, a and b are complex numbers, r is a positive integer. $\lambda_i = ac_i$ where a and c_i are not simultaneously zero, and $i = 1, .., r$.

Further specializing the parameters and using (5), (6) reduces to

$$[D(ax^v + b) - \lambda_1]...[D(ax^v + b) - \lambda_r]\{(ax^v + b)^{\alpha}\ {}_p\overline{F}_q[^{(a_j,1;\alpha_j)_{1,p}}_{(b_j,1;\beta_j)_{1,q}}; z(ax^v + b)]\}$$

$$= a^r\prod_{j=1}^{r}(\alpha - c_j + 1)(ax^v + b)^{\alpha}\ {}_{p+r}\overline{F}_{q+r}[/^{(\alpha-c_j+2,1;1)_{1,r},\,(a_j,1;\alpha_j)_{1,p}}_{(1+\alpha_j-c_j,1;1)_{1,r},\,(b_j,1;\beta_j)_{1,q}}] \tag{11}$$

provided $h > 0$,a and b are complex numbers, r is a positive integer.$\lambda_i = ac_i$ where a and c_i are not simultaneously zero, and $i = 1, .., r$.

When $r = 1$, (11) reduces to

$$[D(ax^v + b) - \lambda]\{(ax^v + b)^{\alpha}.{}_p\overline{F}_q[^{(a_j,1;\alpha_j)_{1,p}}_{(b_j,1;\beta_j)_{1,q}}; z(ax^v + b)]\}$$

$$= a(\alpha - c + 1)(ax^v + b)^{\alpha}_{p+1}\overline{F}_{q+1}[^{(\alpha-c+2,1;1),\,(a_j,1;\alpha_j)_{1,p}}_{(1+\alpha-c,1;1),\,(b_j,1;\beta_j)_{1,q}}] \tag{12}$$

where a and b are complex numbers, $\lambda = ac$ where a and c are not simultaneously zero.

In (6), (7), (8) *and* (9) take $\alpha_j = 1$, for $j = 1, ..., n$ and $\beta_j = 1$ for $j = m + 1, ..., q$ to obtain the corresponding formulas for H-function of Fox as

$$[D(ax^v + b) - \lambda_1]...[D(ax^v + b) - \lambda_r]\{(ax^v + b)^{\alpha}H^{m,n}_{p,q}[z(ax + b)^h/^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}]\}$$

$$= a^r(ax^v + b)^{\alpha}H^{m,n+r}_{p+r,q+r}[z(ax + b)^h/^{(c_j-\alpha-1)_{1,r},\,(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q},\,(c_j-\alpha,h)_{1,r}}] \tag{13}$$

provided $h > 0$, a and $b$are complex numbers, r is a positive integers. $\lambda_i = ac_i$ where a and c_i are not simultaneously zero, and $i = 1, .., r$.

$$[D(ax^v+b)-\lambda][D(ax^v+b)-\lambda+k]...[D(ax^v+b)-\lambda+(r-1)k]\{(ax^v+b)^{\alpha d+c-1}$$

$$.\overline{H}_{p,q}^{m,n}[z(ax^v+b)^h d/^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}]\}$$

$$= a^r d^r (ax^v+b)^{\alpha d+c-1}\, \overline{H}_{p+1,q+1}^{m,n+1}[z(ax^v+b)^{hd}/^{(1-r-\alpha,),\,(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q},\,(1-\alpha,h)}] \tag{14}$$

provided $d \neq 0, h > 0$, a and b are complex number, r is a positive integer, $\lambda = ac, k = ad$, a and c are not simultaneously zero.

$$[D(ax^v+b)-\lambda]^r\{(ax^v+b)^{\alpha+c-1}\, H_{p,q}^{m,n}[z(ax^v+b)^h/^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}]\}$$

$$= a^r (ax^v+b)^{\alpha+c-1}\, H_{p+r,q+r}^{m,n+r}[z(ax^v+b)^h/^{(-\alpha,h)_{1,r},(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q},(1-\alpha,h)_{1,r}}] \tag{15}$$

provided $h > 0$, a and b are complex numbers, r is a positive integer.

$$[D(ax^v+b)^r]\{(ax^v+b)^{\alpha-1}\, H_{p,q}^{m,n}[z(ax^v+b)^h/^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}]\}$$

$$= a^r (ax^v+b)^{\alpha-1}\, H_{p+r,q+r}^{m,n+r}[z(ax^v+b)^h/^{(-\alpha,h)_{1,r},\,(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q},\,(1-\alpha,h)_{1,r}}] \tag{16}$$

provided $h > 0$, a and b are complex numbers, r is a positive integer.

In(12), put $\alpha_j = \beta_j = 1$ to get the following result-

$$[D(ax^v+b)-\lambda]\{(ax^v+b)^{\alpha}\, {}_pF_q[^{(a_j)_{1,p}}_{(b_j)_{1,q}}; z(ax^v+b)]\}$$

$$= a(\alpha-c+1)(ax^v+b)^{\alpha}_{p+1}F^{q+1}[/^{(\alpha-c+2,1;1),\,(a_j)_{1,p}}_{(1+\alpha-c),\,(b_j)_{1,q}}] \tag{17}$$

provided a and b are complex numbers, $\lambda = ac$, a and c are not simultaneously zero.

By specializing the parameters in (17) a number of other results involving various otherspecial functions can be obtain.

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